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Conditional Lyapunov exponent criteria in terms of ergodic theory

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The conditional Lyapunov exponent is defined for investigating chaotic synchronization, in particular complete synchronization and generalized synchronization. We find that the conditional Lyapunov exponent is expressed as a formula in terms of ergodic theory. Dealing with this formula, we find what factors characterize the conditional Lyapunov exponent in chaotic systems.
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Subject Index A30, A33

1. Introduction

The conditional Lyapunov exponent is defined for investigating chaotic synchronization (Refs. [1–14]), in particular complete synchronization (CS) and generalized synchronization (GS). Transitions from desynchronization to synchronization of trajectories occur when the conditional Lyapunov exponent changes from positive to negative (Refs. [15–17]). Although it is widely known that chaotic synchronization occurs in many systems, it is not clearly known why the conditional Lyapunov exponent changes. For example, it has not completely been clarified why CS occurs in chaotic systems. A report (Ref. [10]) showed that an external forcing input in CS may change the dynamical system to another one. The forcing input changes the balance between phase contraction and expansion, and CS occurs when such a contraction dominates. Although the explanation is well considered, there could be another reason why the conditional Lyapunov exponent may change. Furthermore, they have focused on the mean of external forcing inputs for CS. The study [11] showed that they got the transversal Lyapunov exponent by changing the noise distribution. This may be important. However we expect that more precise information on phase space will lead to more relevant analysis. Therefore, we would like to clarify the relation between the conditional Lyapunov exponent and chaotic synchronization in CS.

We have two claims through this research. The first one is about a formula to express the conditional Lyapunov exponent in terms of ergodic theory, and the second is what factors influence the conditional Lyapunov exponent in chaotic dynamical systems. Although existing research offered some mechanisms for chaotic synchronizations by focusing on mean amplitudes or variances of common input signals (see, e.g., Refs. [12,13]), we find that such mean amplitudes or variances of common input signals are not imperative. Instead, we reveal that a *distribution characteristic* of input signals is the most important factor. We also reveal that this characteristic determines transitions between synchronization and desynchronization, and this does not depend on whether input signals

are chaotic or noisy. Thus, although the second claim can easily be derived from the first one, we emphasize that our second claim is physically important.

In Sect. 2, we describe the definition of conditional Lyapunov exponent in chaotic systems. Furthermore, we describe two main claims in our research.

In Sects. 3–5, we construct solvable chaotic dynamical systems and, by analysis, confirm our claims for such systems.

In Appendix A, we summarize a short introduction to ergodic theory.

2. Definition and main claim

To discuss the conditional Lyapunov exponent, we consider the following simple one-dimensional unidirectional coupling system:

$$x(t+1) = f(x(t)) + \xi(t) \equiv \psi(x(t), \xi(t)), \quad (1)$$

where t is time, $x(t)$ is a response, f is a chaotic mapping, and $\xi(t)$ is an external forcing driver. We define $P_X(X)$ as the probability distribution of variables X for the dynamical system, and also define \mathbb{R}_X as the range in which variables X are defined.

We define the CS of the system as follows (Refs. [8–11,15]). We consider two different trajectories $x_1(t)$ and $x_2(t)$ in Eq. (1) with different initial points. To judge whether the response system exhibits CS, we introduce the conditional Lyapunov exponent. The occurrence of CS implies that the difference between $x_1(t)$ and $x_2(t)$ decreases as time increases. The CS is said to occur when the following condition for synchronization error $e'(t) \equiv |x_2(t) - x_1(t)|$ is satisfied for almost all initial points:

$$\lim_{t \rightarrow \infty} e'(t) = 0. \quad (2)$$

Here $|\dots|$ expresses the Euclidean norm of the argument. The interpretation of Eq. (2) is that the state in the large t limit is independent of any initial state for almost all initial points. The infinitesimal synchronization error to the projection to the x -axis $e(T)$ in system (1) is defined as

$$e(T) := \prod_{t=0}^T \left| \frac{\partial \psi(x, \xi)}{\partial x} \right|_{x=x(t)} e(0). \quad (3)$$

When CS occurs, the equation $\lim_{t \rightarrow \infty} e'(t) = \lim_{T \rightarrow \infty} |e(T)| = 0$ is satisfied. It is noted here that $\xi(t)$ in Eq. (1) affects the time evolution of $x(t)$ and contributes to $e(T)$.

The conditional Lyapunov exponent for variables x is defined by

$$\lambda := \lim_{T \rightarrow \infty} \frac{1}{T} \ln \frac{|e(T)|}{|e(0)|}.$$

Clearly λ expresses the stability of the state $x_1(t) = x_2(t)$. In this paper, the CS is said to occur when $\lambda < 0$, and we do not use Eq. (2) directly. The conditional Lyapunov exponent exists when system (1) is ergodic with respect to x and ξ according to Oseledets' multiplicative ergodic theorem for autonomous dynamical systems (Ref. [18]).

Here, we assume that system (1) can be seen as a two-dimensional autonomous dynamical system of x and ξ . When system (1) is ergodic with respect to the absolutely continuous invariant measure

(physical measure) with respect to x and ξ , the conditional Lyapunov exponent λ is expressed as the ensemble average:

$$\begin{aligned}\lambda &= \lim_{T \rightarrow \infty} \frac{1}{T} \ln \frac{|e(T)|}{|e(0)|} \\ &= \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} P(x, \xi) \ln \left| \frac{\partial \psi(x, \xi)}{\partial x} \right| d\xi dx,\end{aligned}\quad (4)$$

where $P(x, \xi)$ is the invariant distribution of system (1).

We can generalize Eq. (4) to higher-dimensional dynamical systems. We define the following two-dimensional dynamical system with an external forcing input:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t)) + \xi(t) \equiv \psi_1(x_1(t), x_2(t), \xi(t)) \\ x_2(t+1) = f_2(x_1(t), x_2(t)) + \xi(t) \equiv \psi_2(x_1(t), x_2(t), \xi(t)). \end{cases}\quad (5)$$

If the two-dimensional dynamical system (5) is ergodic with respect to the absolutely continuous invariant measure in terms of x_1 and x_2 variables, the conditional Lyapunov exponent $\lambda_{ij} (i = 1, 2, j = 1, 2)$, which is defined by the infinitesimal synchronization error for ψ_j to the projection to the x_i -axis, is defined by

$$\begin{aligned}\lambda_{ij} &= \lim_{T \rightarrow \infty} \frac{1}{T} \ln \frac{|e_{\psi_j}(T)|}{|e_{x_i}(0)|} \\ &= \int_{\mathbb{R}_{x_1}} \int_{\mathbb{R}_{x_2}} \int_{\mathbb{R}_\xi} P(x_1, x_2, \xi) \ln \left| \frac{\partial \psi_j(x_1, x_2, \xi)}{\partial x_i} \right| d\xi dx_1 dx_2.\end{aligned}$$

In what follows, however, we assume that system (1) has a one-dimensional limiting distribution given by the absolutely continuous invariant ergodic measure $\mu(dx) = P_x(x) dx$ ($x \in \mathbb{R}_x$) with some P_x , and $P(x, \xi)$ is a continuous density function in x and ξ satisfying $\int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} P(x, \xi) d\xi dx = 1$ for simplicity.

Then, we give our two main claims: First, the conditional Lyapunov exponent is expressed by the equation

$$\lambda = \tilde{\lambda}, \quad (6)$$

where

$$\lambda = \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} P(x, \xi) \ln \left| \frac{\partial \psi(x, \xi)}{\partial x} \right| d\xi dx, \quad (7)$$

$$\tilde{\lambda} = \int_{\mathbb{R}_\xi} P_\xi(\xi_0) \lambda_\Psi(\xi_0) d\xi_0. \quad (8)$$

Note that although ξ_0 is originally a constant value, we consider the *distribution* of ξ_0 in the above equation. Here $\lambda_\Psi(\xi_0)$ is defined as follows: First, by calculating the Lyapunov exponent $\lambda_\Psi(\xi_0)$ for the auxiliary system,

$$y(t+1) = \Psi_{\xi_0}(y(t)) \quad (9)$$

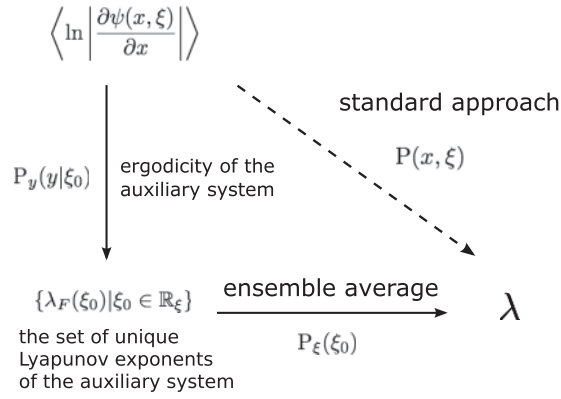


Fig. 1. Our approach for the conditional Lyapunov exponent.

with $\Psi_{\xi_0}(y) = f(y) + \xi_0$. Then, by changing ξ_0 continuously, we have the set $\{\lambda_\Psi(\xi_0) \mid \xi_0 \in \mathbb{R}_\xi\}$. Note that the functions Ψ_{ξ_0} are one-dimensional functions with the parameters ξ_0 distributed according to $P(\xi)$, while $\psi(y, \xi_0)$ is just a two-dimensional function that is different from $\Psi_{\xi_0}(y)$.

Equation (6) implies that the conditional Lyapunov exponent λ is expressed as the *ensemble average of the set of unique Lyapunov exponents* $\{\lambda_\Psi(\xi_0) \mid \xi_0 \in \mathbb{R}_\xi\}$ provided the ergodicity in the one-dimensional dynamical system (9). This is the first main claim in our research.

Then, we show how we get this claim provided we assume ergodicity in system (1). The Lyapunov exponent $\lambda_\Psi(\xi_0)$ can be obtained by the following equation with the ergodicity in system (9):

$$\lambda_\Psi(\xi_0) = \int_{\mathbb{R}_y} P_y(y|\xi_0) \ln \left| \frac{d\Psi_{\xi_0}(y)}{dy} \right| dy, \quad (10)$$

where $P(y|\xi_0)$ is the unique *conditional probability distribution* in dynamical system (9) with each constant value ξ_0 . To obtain Eq. (8) we use the *marginalization* of random variables a and b in probability theory:

$$P(a, b) = P(b)P(a|b). \quad (11)$$

Then, substituting Eq. (11) into Eq. (7), we have

$$\begin{aligned} \lambda &= \int_{\mathbb{R}_y} \int_{\mathbb{R}_\xi} P_\xi(\xi_0) P_y(y|\xi_0) \ln \left| \frac{d\Psi_{\xi_0}(y)}{dy} \right| d\xi_0 dy \\ &= \int_{\mathbb{R}_\xi} P_\xi(\xi_0) \left(\int_{\mathbb{R}_y} P_y(y|\xi_0) \ln \left| \frac{d\Psi_{\xi_0}(y)}{dy} \right| dy \right) d\xi_0 \\ &= \int_{\mathbb{R}_\xi} P_\xi(\xi_0) \lambda_\Psi(\xi_0) d\xi_0 = \tilde{\lambda}. \end{aligned} \quad (12)$$

Note that $\lambda = \tilde{\lambda}$ is satisfied when the invariant distribution $P_y(y|\xi)$ exists. This is satisfied when system (9) is ergodic for any ξ_0 . Therefore, our first claim (Eq. (6)) is derived from the mixing property of Eq. (1) and the ergodicity of ξ and system (9). Figure 1 illustrates this concept. Here, $\langle \dots \rangle$ expresses the ensemble average with respect to the ergodic invariant measure.

Note that if we have two-dimensional dynamical systems, the conditional Lyapunov exponent λ_{ij} ($i = 1, 2, j = 1, 2$), which is defined by the infinitesimal synchronization error to the projection to

x_i -axis, is expressed as

$$\lambda_{ij} = \int_{\mathbb{R}_{x_1}} \int_{\mathbb{R}_{x_2}} \int_{\mathbb{R}_{\xi}} P_{\xi}(\xi_0) P_{x_1, x_2}(x_1, x_2 | \xi_0) \ln \left| \frac{\partial \Psi_{j\xi_0}(x_1, x_2)}{\partial x_i} \right| d\xi_0 dx_1 dx_2.$$

As for the second claim, with the first one, we can derive that the conditional Lyapunov exponent in system (1) is characterized by only two factors: a dynamical system and a distribution of external forcing input.

3. Example 1 for the first claim

In order to confirm the first claim, we would like to show its relevance by analyzing solvable chaotic synchronization systems (Refs. [19,20]). Here, a solvable chaotic synchronizing system is such that an invariant measure, the conditional Lyapunov exponent, and the threshold between synchronization and desynchronization are *analytically* obtained for a unidirectional coupling system.

3.1. Definition of our system

As for the first claim, we need three steps to show it. First, we analytically calculate the conditional Lyapunov exponent λ in our solvable chaotic dynamical system in accordance with definition (4). Second, we analyze the auxiliary dynamical system similar to Eq. (9), and get the exponent $\tilde{\lambda}$ in accordance with Eqs. (9)–(12). Third, we confirm that the results obtained by the two types of analytical methods coincide.

We study the dynamical system

$$x(t+1) = f(x(t)) + \varepsilon \zeta(t), \quad (13)$$

where ε is a coupling parameter, and $\varepsilon \zeta(t)$ corresponds to an external forcing input $\xi(t)$. First we prepare the solvable chaotic dynamical system (Refs. [21–23])

$$\begin{aligned} x(t+1) &= f(x(t)) + \varepsilon \zeta(t) \equiv \phi(x(t), \zeta(t)), \\ f(x(t)) &= g(x(t)) \equiv \frac{1}{2} \left(x(t) - \frac{1}{x(t)} \right), \\ \zeta(t+1) &= g(\zeta(t)), \end{aligned} \quad (14)$$

where the function g is associated with the double-angle formula given by $\cot 2\theta = \frac{1}{2} \left(\cot \theta - \frac{1}{\cot \theta} \right)$ (Ref. [20]). The mapping associated with g is a chaotic mapping that has the mixing property (Ref. [20]), and its Lyapunov exponent is $\ln 2$. The invariant measure of system $x(t+1) = g(x(t))$ is the standard Cauchy distribution,

$$\mu(dx) = C(x; 0, 1) dx,$$

where $C(x)$ is a Cauchy distribution defined as

$$C(x; c, \gamma) \equiv \frac{\gamma}{\pi \{(x - c)^2 + \gamma^2\}},$$

with c a median and γ a scale parameter. Therefore, the distribution of variable x in the dynamical system $x(t+1) = g(x(t))$ follows $C(x; 0, 1)$ (Ref. [20]), and the distribution of the external forcing ξ also follows $C(x; 0, 1)$. This is our first solvable chaotic dynamical system. We see that the conditional Lyapunov exponent changes when the coupling parameter is varied. We calculate the conditional Lyapunov exponent of this dynamical system by definition (4) in accordance with the first step.

3.2. Conventional ergodic theoretical approach

We express the conditional Lyapunov exponent of system (14) as $\lambda_g(\varepsilon)$ since it will turn out that the conditional Lyapunov exponent crucially depends on ε . The conditional Lyapunov exponent $\lambda_g(\varepsilon)$ is expressed as

$$\begin{aligned}\lambda_g(\varepsilon) &= \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} P_{x,\xi} \ln \left| \frac{\partial \phi(x, \xi)}{\partial x} \right| d\xi dx \\ &= \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} P_{x,\xi}(x) P_\xi(\xi) d\xi \ln \left| \frac{1}{2} \left(1 + \frac{1}{x^2} \right) \right| dx \\ &= \int_{\mathbb{R}_x} P_{x,\xi}(x) \ln \left| \frac{1}{2} \left(1 + \frac{1}{x^2} \right) \right| dx \\ &= \int_{\mathbb{R}_x} P_{x,\varepsilon}(x) \ln \left| \frac{1}{2} \left(1 + \frac{1}{x^2} \right) \right| dx,\end{aligned}$$

where $P_{x,\xi}(x)$ is the invariant distribution for the variable x in the superposed dynamical system $x(t+1) = g(x(t)) + \varepsilon \zeta(t)$ with a parameter ε . Note that the superposed distribution is expressed as $P_{x,\varepsilon}(x)$ since ζ is given and the external forcing ξ is dependent only on the coupling parameter ε .

Then, we first calculate the invariant distribution $P_{x,\varepsilon}(x)$, second the conditional Lyapunov exponent $\lambda_g(\varepsilon)$, and third the threshold between synchronization and desynchronization analytically.

We can calculate the invariant distribution $P_{x,\varepsilon}(x)$ by using the following three properties: (property 1) the mixing property for the mapping g , (property 2) the property of preserving the form of Cauchy distributions in terms of the Perron–Frobenius equation (PF equation) for g , and (property 3) the property of the stable property for Lévy distributions. Then showing these three properties, we explain how they play roles to get $P_{x,\varepsilon}(x) dx$.

As to property 1, it is already known that the mapping g has the mixing property (Ref. [20]).

As for property 2, this property implies that g is the mapping that changes an input Cauchy distribution into *another* Cauchy distribution with a *different* median and scale parameter. We consider the PF equation for the $z = g(x)$ where the input variable x follows $C(x; c, \gamma)$. Since g is a two-to-one mapping, $P_z(z)$ satisfies the following PF equation (probability preservation relation):

$$P_z(z) |dz| = C(x_1; c, \gamma) |dx_1| + C(x_2; c, \gamma) |dx_2|,$$

where x_1 and x_2 ($x_1 > x_2$) are the solutions of the quadratic equation $z = g(x)$. They satisfy

$$\begin{cases} x_1 + x_2 = 2z, \\ x_1 x_2 = -1. \end{cases}$$

With these relations, the probability distribution $P_z(z)$ is obtained as the following rescaled Cauchy distribution:

$$P_z(z) = C(z; c', \gamma'),$$

where $c' = c(\gamma^2 + c^2 - 1)/2(\gamma^2 + c^2)$ and $\gamma' = \gamma(\gamma^2 + c^2 + 1)/2(\gamma^2 + c^2)$.

As to property 3, this stable property is already widely known. When the distributions of two independent variables X_1 and X_2 obey a Lévy distribution family, the distribution of the variable

$aX_1 + bX_2$ ($a, b \in \mathbb{R}$) also obeys the same family. These three properties play roles in getting the invariant distribution $P_{x,\varepsilon}(x)$.

We can prove that variables x and $\varepsilon\zeta$ in system (14) obey Cauchy distributions respectively with three properties. In Appendix B, we describe these three properties in more detail. By taking into account properties 2 and 3, the distribution of the variable x is changed to a Cauchy distribution with a different median and scale parameter in every iteration if the initial input follows a Cauchy distribution. Hence, we get the following self-consistent recurrence equations about a median c and a scale parameter γ per iteration:

$$\begin{cases} c(t+1) &= \frac{c(t)(\gamma(t)^2 + c(t)^2 - 1)}{2(\gamma(t)^2 + c(t)^2)}, \\ \gamma(t+1) &= \frac{\gamma(t)(\gamma(t)^2 + c(t)^2 + 1)}{2(\gamma(t)^2 + c(t)^2)} + |\varepsilon|. \end{cases} \quad (15)$$

We get the following convergence values c^* and γ^* as the stable fixed point of Eq. (15) for $t \rightarrow \infty$:

$$c_g^* = 0 \quad \gamma_g^* = |\varepsilon| + (\varepsilon^2 + 1)^{1/2}.$$

Hence, we analytically obtain $P_{x,\varepsilon}(x)$ as the fixed point of the recurrence equation Eq. (15):

$$P_{x,\varepsilon}(x) = C(x; 0, \gamma_g^*).$$

We should emphasize the following: Although we assume that the initial input follows a Cauchy distribution in the above analysis, we do not need any restriction for the distribution of an initial point for system (1) because of the mixing property (property 1). Therefore, the invariant distribution $P_{x,\varepsilon}(x)$ is always obtained regardless of any initial points.

With $P_{x,\varepsilon}(x)$, we can calculate the conditional Lyapunov exponent $\lambda_g(\varepsilon)$ as

$$\begin{aligned} \lambda_g(\varepsilon) &= \int_{\mathbb{R}_x} P_{x,\varepsilon}(x) \ln \left| \frac{1}{2} \left(1 + \frac{1}{x^2} \right) \right| dx \\ &= \int_{\mathbb{R}} \frac{\gamma_g^*}{\pi(x^2 + \gamma_g^{*2})} \ln \left| \frac{1}{2} \left(1 + \frac{1}{x^2} \right) \right| dx \\ &= 2 \ln \left(\frac{\gamma_g^* + 1}{\gamma_g^*} \right) - \ln 2 \\ &= 2 \ln \left((\varepsilon^2 + 1)^{1/2} - |\varepsilon| + 1 \right) - \ln 2. \end{aligned}$$

Note that $\lambda_g(0) = \ln 2$. We can also get the threshold ε_g^* between the synchronization and desynchronization. This is the solution of $\lambda_g(\varepsilon_g^*) = 0$, which is

$$\varepsilon_g^* = 1.$$

Figure 2 illustrates that these analytical results coincide with the results of numerical simulation with the initial condition $x_0 = \sqrt{2}$ and $\xi_0 = \sqrt{3}$.

3.3. Proposing an ergodic theoretical approach

As above, we can calculate $\lambda_g(\varepsilon)$ by Eq. (4). The invariant distribution $P(x, \xi)$ is expressed as

$$P(x, \xi) = \frac{\gamma_g^*}{\pi(x^2 + \gamma_g^{*2})} \frac{|\varepsilon|}{\pi(\xi^2 + \varepsilon^2)},$$

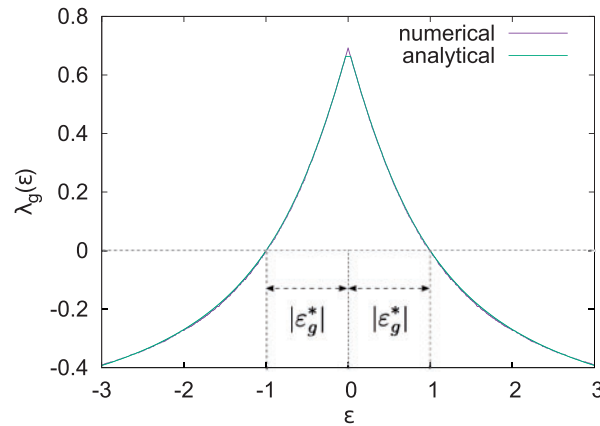


Fig. 2. Conditional Lyapunov exponent $\lambda_g(\varepsilon)$.

because $P_\xi(\xi)$ is derived from property 3 for the variable ζ that obeys $C(\zeta; 0, 1)$. Then in accordance with the second step, we would like to calculate $\tilde{\lambda}_g(\varepsilon)$ by Eqs. (9)–(12).

We prepare the following dynamical system by Eq. (9):

$$\begin{cases} y(t+1) = G_{\xi_0}(y(t)), \\ G_{\xi_0}(y) = g(y) + \xi_0. \end{cases}$$

We first need to confirm that this system is ergodic for every ξ_0 , which is the sufficient condition for the conditional distribution $P_y(y|\xi_0)$ to exist for every ξ_0 . Hence, we sufficiently calculate three factors: the conditional distribution $P_y(y|\xi_0)$, the unique Lyapunov exponents $\lambda_\Psi(\xi_0)$, and the distribution of external forcing inputs $P_\xi(\xi_0)$.

First we calculate the conditional distribution $P_y(y|\xi_0)$. The invariant measure $P_y(y|\xi_0) dx$ is also obtained with properties 1 to 3.

In the same way as getting $P_{x,\xi}(x)$, we can prove that $P_y(y|\xi_0)$ also follows a Cauchy distribution. Hence, we get the following self-consistent recurrence equations about a median c and a scale parameter γ :

$$\begin{cases} c(t+1) = \frac{c(t)(\gamma(t)^2 + c(t)^2 - 1)}{2(\gamma(t)^2 + c(t)^2)} + \xi_0, \\ \gamma(t+1) = \frac{\gamma(t)(\gamma(t)^2 + c(t)^2 + 1)}{2(\gamma(t)^2 + c(t)^2)}. \end{cases}$$

We get the following convergence values \hat{c} and $\hat{\gamma}$ as the stable fixed point for $t \rightarrow \infty$:

$$\begin{cases} \hat{c} = \xi_0, \\ \hat{\gamma} = (1 - \xi_0^2)^{1/2} \end{cases} \quad (\text{if } |\xi_0| < 1),$$

$$\begin{cases} \hat{c} = \xi_0 + \text{sgn}(\xi_0)(\xi_0^2 - 1)^{1/2}, \\ \hat{\gamma} = 0 \end{cases} \quad (\text{if } |\xi_0| \geq 1),$$

$$\text{and } \text{sgn}(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 > 0, \\ -1 & \text{if } \xi_0 < 0. \end{cases}$$

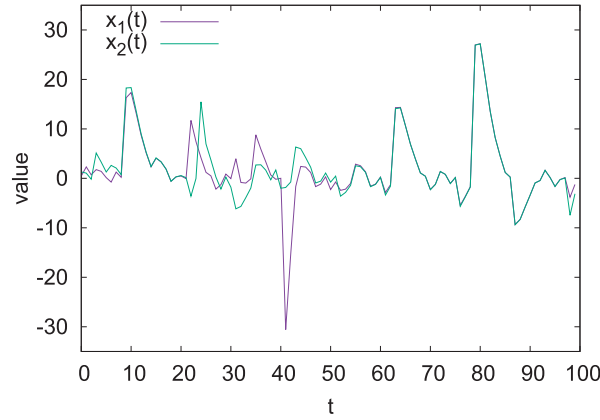


Fig. 3. Partial synchronization in system (14) with $\varepsilon = 0.8$.

Thus, we obtain the conditional distribution

$$P_y(y|\xi_0) = \begin{cases} C(y; \xi_0, (1 - \xi_0^2)^{1/2}) & (\text{if } |\xi_0| < 1), \\ C(y; \xi_0 + \text{sgn}(\xi_0)(\xi_0^2 - 1)^{1/2}, 0) & (\text{if } |\xi_0| \geq 1). \end{cases}$$

As for unique Lyapunov exponents $\lambda_\Psi(\xi_0)$, we can calculate them as

$$\begin{aligned} \lambda_G(\xi_0) &= \int_{\mathbb{R}_y} P_y(y|\xi_0) \ln \left| \frac{1}{2} \left(1 + \frac{1}{y^2} \right) \right| dy \\ &= \begin{cases} \ln(1 + (1 - \xi_0^2)^{1/2}) & (\text{if } |\xi_0| < 1), \\ \ln \left\{ 1 + \frac{1}{(\xi_0 + \text{sgn}(\xi_0)(\xi_0^2 - 1)^{1/2})^2} \right\} - \ln 2 & (\text{if } |\xi_0| \geq 1). \end{cases} \end{aligned}$$

Thus, we can calculate the exponent $\tilde{\lambda}_g(\varepsilon)$ according to Eq. (12):

$$\begin{aligned} \tilde{\lambda}_g(\varepsilon) &= \int_{\mathbb{R}_\xi} P_\xi(\xi_0) \lambda_G(\xi_0) d\xi_0 \\ &= \int_{\mathbb{R}} \frac{|\varepsilon|}{\pi (\xi_0^2 + \varepsilon^2)} \lambda_G(\xi_0) d\xi_0 \\ &= 2 \ln \left((\varepsilon^2 + 1)^{1/2} - |\varepsilon| + 1 \right) - \ln 2 \\ &= \lambda_g(\varepsilon). \end{aligned}$$

As above, we confirm that our first main claim is certainly satisfied. This claim signifies that a conditional Lyapunov exponent is expressed as the ensemble average of the set of unique Lyapunov exponents of the auxiliary dynamical system. Figure 3 illustrates that two different trajectories with different initial points ($x_1(0) = 0.46$, $x_2(0) = 1.3$) partially synchronize. The coupling parameter ε is 0.8. Note that although the conditional Lyapunov exponent is positive in Fig. 3, the partial synchronization occurs since it is the averaged factor of unique Lyapunov exponents. When the conditional Lyapunov exponent is negative, the infinitesimal synchronization error converges to 0 for $t \rightarrow \infty$ (see Fig. 4).

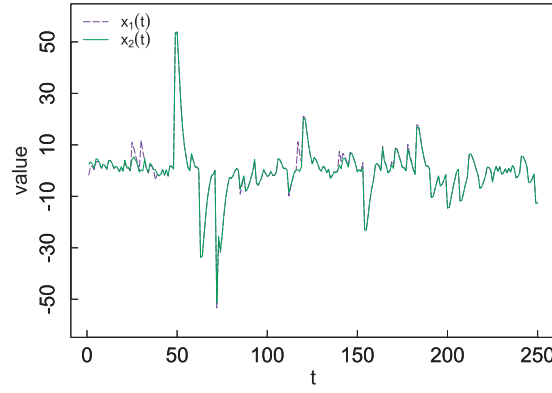


Fig. 4. Chaotic synchronization in system (14) with $\varepsilon = 1.05$ (and $x_1(0) = -\sqrt{2}$, $x_2(0) = \sqrt{5}$).

4. Example 2 for the first claim

Our analysis is not restricted to the above example. We consider another solvable chaotic dynamical system, and can also confirm that the conditional Lyapunov exponent in the system is obtained by Eq. (6). This dynamical system is defined as

$$\begin{aligned} x(t+1) &= f(x(t)) + \varepsilon \zeta(t), \\ f(x(t)) &= h(x(t)) \equiv \frac{2x(t)}{1-x(t)^2}, \\ \zeta(t+1) &= h(\zeta(t)), \\ H_{\xi_0}(y) &= h(y) + \xi_0. \end{aligned}$$

The chaotic mapping h is associated with the double formula $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ (Ref. [20]). Its Lyapunov exponent is $\ln 2$, and the invariant measure obeys the standard Cauchy distribution. Then, the invariant distribution $P_{x,\xi}(x)$, the conditional Lyapunov exponent $\lambda_h(\varepsilon)$, and the threshold ε_h^* are

$$\begin{aligned} P_{x,\xi}(x) &= C(x; 0, \gamma_h^*), \\ \lambda_h(\varepsilon) &= 2 \ln \left(\frac{\gamma_h^* + 1}{\gamma_h^{*2} + 1} \right) + \ln 2, \\ \varepsilon_h^* &= 0.78 \dots, \end{aligned}$$

where γ_h^* is the positive number that satisfies

$$\gamma_h^{*3} - |\varepsilon| \gamma_h^{*2} - \gamma_h^* - |\varepsilon| = 0.$$

5. Example for the second claim

Here, as for the second claim, the conditional Lyapunov exponent for system (1) is characterized by only two factors: a dynamical system associated with f and a distribution of external forcing input ξ .

To show this claim we consider six different dynamical systems. Each of them has a different chaotic mapping f or different system for ξ :

$$f(x) = \begin{cases} g(x) = \frac{1}{2} \left(x - \frac{1}{x} \right), \\ h(x) = \frac{2x}{1-x^2}, \end{cases}$$

Table 1. Relations between combinations of f , ξ , and the conditional Lyapunov exponents.

f	Generation mechanism of ζ	Lyapunov exponent of f	Invariant distribution of system			Property of time series in external forcing	
			$x(t+1) = f(x(t))$	$P_\xi(\xi)$	$P_{x,\xi}(x)$	input ξ	λ
g	g	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_g^*)$	chaotic	$\lambda_g(\varepsilon)$
g	h	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_g^*)$	chaotic	$\lambda_g(\varepsilon)$
g	Crand(0,1)	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_g^*)$	random	$\lambda_g(\varepsilon)$
h	g	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_h^*)$	chaotic	$\lambda_h(\varepsilon)$
h	h	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_h^*)$	chaotic	$\lambda_h(\varepsilon)$
h	Crand(0,1)	$\ln 2$	$C(x; 0, 1)$	$C(\xi; 0, \varepsilon)$	$C(x; 0, \gamma_h^*)$	random	$\lambda_h(\varepsilon)$

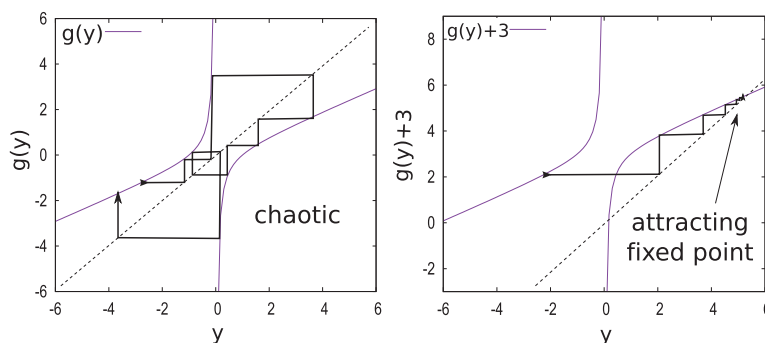


Fig. 5. Attracting fixed point is generated by changing ξ_0 in $G_{\xi_0}(y)$.

$$\begin{cases} \zeta(t+1) = g(\zeta(t)), \\ \zeta(t+1) = h(\zeta(t)), \\ \zeta = \text{Crand}(0, 1), \end{cases}$$

where $\text{Crand}(c, \gamma)$ is a set of random numbers that follow $C(\zeta; c, \gamma)$. The algorithm to get $\text{Crand}(c, \gamma)$ is

$$\text{Crand}(0, 1) = \tan\left(\frac{\pi}{2}(U(-1 : 1))\right),$$

where $U(-1 : 1)$ are uniform random numbers on the interval $(-1 : 1)$.

We compare the conditional Lyapunov exponent of these systems with Table 1. As we see from the Table 1, there are some factors that do not influence the conditional Lyapunov exponent, such as the property of time series in external forcing inputs. Then, we show that the conditional Lyapunov exponent is characterized by the original mapping f and the distribution of external forcing input ξ . The original dynamical system is changed to the system that has an attracting fixed point because of the external forcing input. The conditional Lyapunov exponent is changed by the existence of attracting fixed points. Figures 5 and 6 illustrate that the mappings G and H , respectively, are changed to ones that have attracting fixed points by external forcing input. Attractors of systems depend on how attracting fixed points are generated. The created attractor leads to the ξ -dependence of $\lambda_\psi(\xi_0)$. Figures 7 and 8 show the sets of Lyapunov exponents $\lambda_G(\xi_0)$ and $\lambda_H(\xi_0)$, respectively. Hence, the original mappings determine how attracting fixed points are generated. Furthermore, with

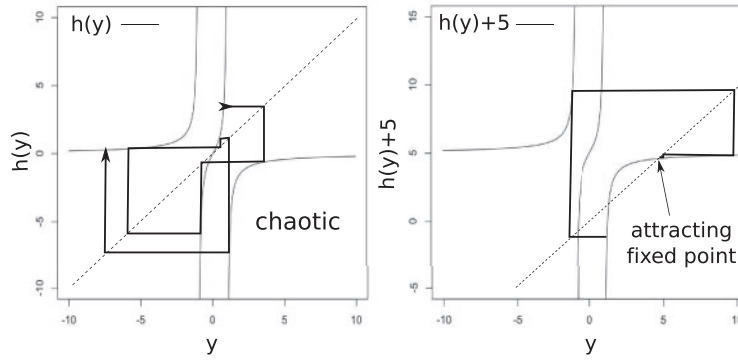


Fig. 6. Attracting fixed point is generated by changing ξ_0 in $H_{\xi_0}(y)$.

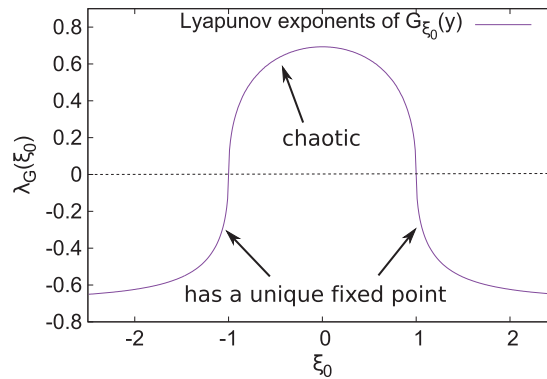


Fig. 7. Unique Lyapunov exponents of $G_{\xi_0}(y)$.

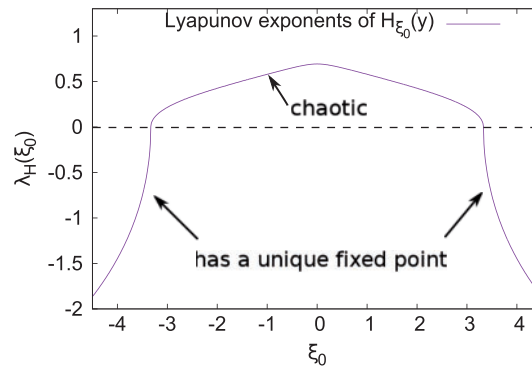


Fig. 8. Unique Lyapunov exponents of $H_{\xi_0}(y)$.

Eq. (6) from the first claim, the conditional Lyapunov exponent is calculated by the set of Lyapunov exponents $\{\lambda_\Psi(\xi_0) \mid \xi_0 \in \mathbb{R}_\xi\}$ and the distribution of the external forcing input P_ξ . As above, the conditional Lyapunov exponent in system (1) is uniquely characterized by only two factors: the original dynamical system and the distribution of external forcing input.

6. Conclusion

We have two main claims. First, the conditional Lyapunov exponent is expressed as Eq. (6) provided the ergodicity in dynamical systems. This yields that the conditional Lyapunov exponent is expressed

as the ensemble average of the set of unique Lyapunov exponents of the auxiliary dynamical system (9). Second, the conditional Lyapunov exponent is characterized by only two factors: an original dynamical system and a distribution of external forcing input. Then, although we consider CS in one-dimensional unidirectionally coupled dynamical systems only, this claim will also hold in multidimensional chaotic systems in terms of ergodic theory.

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Appendix A. Ergodic dynamical systems

In this appendix, a short introduction of ergodic theory is provided. Since this appendix aims to provide clear definitions of terminology used in the main text, it is not comprehensive. For a comprehensive review from a mathematical viewpoint see Refs. [24,25], and from a physical viewpoint see Ref. [26].

Definition A.1 (Dynamical system, Ref. [24]). A dynamical system $(\mathcal{M}, \mu, \varphi_t)$ is a measure-space (\mathcal{M}, μ) equipped with a one-parameter group φ_t of automorphisms (except for spaces of measure zero) of (\mathcal{M}, μ) , φ_t depending measurably on t . Here the parameter t denotes an integer.

Given a dynamical system $(\mathcal{M}, \mu, \varphi_t)$, it follows that $\mu(\varphi_t A) = \mu(A)$, where A is a measurable set, and that φ_t is measurable in $\mathcal{M} \times \mathbb{R}$. In what follows $\mu(\mathcal{M}) = 1$ is assumed.

The following average often appears in physics.

Definition A.2 (Time-average, Ref. [24]). Let $(\mathcal{M}, \mu, \varphi_t)$ be a dynamical system, and f a complex-valued function defined on \mathcal{M} . If there exists the quantity

$$f^*(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\varphi^n x), \quad x \in \mathcal{M}, \quad n \in \mathbb{Z},$$

then $f^*(x)$ is called the time-average of f .

The following average also often appears and is related to the time-average for some dynamical systems.

Definition A.3 (Space-average, Ref. [24]). Let $(\mathcal{M}, \mu, \varphi_t)$ be a dynamical system, and f a complex-valued function defined on \mathcal{M} . If there exists the quantity

$$\bar{f} = \int_{\mathcal{M}} f(x) \mu(dx), \quad x \in \mathcal{M},$$

then \bar{f} is called the space-average of f .

The space-average defined above is called the ensemble average in the main text. The following is used in the main text.

Definition A.4 (Absolutely continuous function with respect to the Lebesgue measure). In Definition A.3, if $\mu(dx)$ is of the form

$$\mu(dx) = \rho(x) dx,$$

then ρ is called an absolutely continuous function with respect to the Lebesgue measure dx .

We are now ready to state the definition of an ergodic system.

Definition A.5 (Ergodic system, Ref. [24]). Let $(\mathcal{M}, \mu, \varphi_t)$ be a dynamical system. If the condition

$$f^*(x) = \bar{f}$$

is satisfied for any integrable function f in the sense of $f \in L_1(\mathcal{M}, \mu)$, then the dynamical system is called an ergodic system.

This states that for an ergodic dynamical system, one can replace the time-average of f with the space-average of it. An example of how to apply this property is Eq. (4) in the main text.

The following property is a stronger property for dynamical systems.

Definition A.6 (Mixing system, Ref. [24]). Let $(\mathcal{M}, \mu, \varphi_t)$ be a dynamical system. If the condition

$$\lim_{t \rightarrow \infty} \mu[\varphi_t A \cap B] = \mu(A) \mu(B)$$

is satisfied for all measurable sets A and B , then the dynamical system is called a mixing system.

Appendix B. Three properties

In this appendix, we describe properties 2 and 3 from Sect. 3 in the main text in more detail. For property 2, we get $P_z(z)$ in Sect. 3.2 as follows:

$$\begin{aligned} P_z(z) &= \frac{\gamma}{\pi((x_1 - c)^2 + \gamma^2)} \left| \frac{1}{dz/dx_1} \right| + \frac{\gamma}{\pi((x_2 - c)^2 + \gamma^2)} \left| \frac{1}{dz/dx_2} \right| \\ &= C(x_1; c, \gamma) \frac{2x_1^2}{x_1^2 + 1} + C(x_2; c, \gamma) \frac{2x_2^2}{x_2^2 + 1} \\ &= \frac{\frac{\gamma(\gamma^2 + c^2 + 1)}{2(\gamma^2 + c^2)}}{\pi \left\{ \left(z - \frac{c(\gamma^2 + c^2 + 1)}{2(\gamma^2 + c^2)} \right)^2 + \left(\frac{\gamma(\gamma^2 + c^2 + 1)}{2(\gamma^2 + c^2)} \right)^2 \right\}} \\ &= C(z; c', \gamma'). \end{aligned}$$

This shows that the mapping g changes the median c and the scale parameter γ of the input Cauchy distribution as:

$$\begin{aligned} \text{median : } c &\rightarrow \frac{c(\gamma^2 + c^2 - 1)}{2(\gamma^2 + c^2)} (\equiv c'), \\ \text{scale parameter : } \gamma &\rightarrow \frac{\gamma(\gamma^2 + c^2 + 1)}{2(\gamma^2 + c^2)} (\equiv \gamma'). \end{aligned}$$

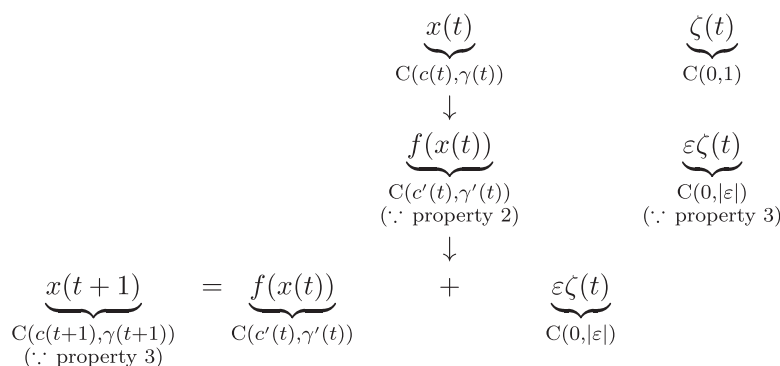


Fig. B1. Propagation of Cauchy distributions from $x(t)$ and $\zeta(t)$ to $x(t+1)$.

We utilize properties 1 to 3 to get Eq. (15). When the variables $x(t)$ in the dynamical system Eq. (14) follow a Cauchy distribution $C(c(t), \gamma(t))$ in the dynamical system Eq. (13), the variables $x(t+1)$ also follow a Cauchy distribution as Fig. B1. Hence, we get the self-consistent recurrence equations (15) about a median c and a scale parameter γ per iteration. This idea is also utilized to get $P_y(y|\xi_0)$ in Sect. 3.3.

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